Department of Pure & Applied Mathematics

Model Answer

Paper: Mechanics-I
Paper Code: **AU-6868**

1(i) For common catenary we know that

$$x = c\log(\sec\psi + \tan\psi) \quad \text{—— (I)}$$

$$s = c\tan\psi \quad \text{—— (II)}$$

and $$y = c\sec\psi \quad \text{—— (III)}$$

Putting (II) and (III) in (I), we have

$$\boxed{x = c\log\frac{y+s}{c}}$$

(ii) Principal moment of inertia: A pair of axes about which the product of inertia vanishes is called principal axes. Moments of inertia about principal axes are called principal moments of inertia.

(iii) Common Catenary: The curve in which a uniform chain or a perfectly flexible string hangs freely suspended from two fixed points is called a common catenary. equation is given by

$$\boxed{s = c\tan\psi}$$

(IV). Impulsive Force:

The force that two colliding bodies exert on one another acts only for short time, giving a bref but strong push. This force is called impulsive force.

(V). Law of Conservation of momentum: "If the net external force acting on a particle vanishes, its linear momentum is conserved"

"If net external torque acting on the system is zero, the total angular momentum of the system is conserved"

(VI). In simple pendulum mass of string is taken as mass-less but in the case of compound pendulum ~~mass~~ of the rod ~~is not~~ have mass. In case of compound pendulum periods depends on its moment of inertia I around the pivot point.

(vii). Unstable equilibrium: If a body move further away so that the body never returns to its original position of equilibrium, in this case equilibrium is said to be unstable.

(viii) We know that moment of inertia of a lamina about an axis making an angle θ with OX is given by

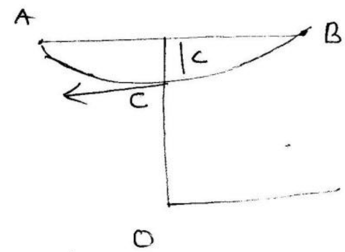
$$A \cos^2\theta + B \sin^2\theta - 2F \sin\theta \cos\theta$$

$$\Rightarrow \quad A \cos^2\theta + B \sin^2\theta - 2F \sin\theta \cos\theta = \frac{k}{r^2} \text{ (say)}$$

$$\Rightarrow \quad Ax^2 + By^2 - 2Fxy = K \quad \text{——— (1)}$$

For different values of k, we get family of concentric ellipses called momental ellipse.

2 = Length of the Chain $= 2l$



$AB = 2a$

and l is little greater than a.

Tension of the chain at lowest point

$$T_0 = wc \quad \text{——— (1)}$$

We know that for common Catenary

$$s = c \sinh \frac{x}{c}$$

For the point B, $s = l$, and $x = a$

This implies

$$l = c \sinh \frac{a}{c}$$

$$l = c \left[\frac{a}{c} + \frac{a^3}{3!\,c^3} + \frac{a^5}{5!\,c^5} + \cdots \right]$$

$$\Rightarrow \quad l = c \left[\frac{a}{c} + \frac{a^3}{6c^3} \right]$$

$$l = a + \frac{a^3}{6c^2}$$

$$\Rightarrow \quad c = \sqrt{\frac{a^3}{6(l-a)}}$$

putting this value in (1), we have

$$T_0 = w\sqrt{\frac{a^3}{6(l-a)}}$$

This shows that tension the chain is approximately equal to the weight of the length $\sqrt{\frac{a^3}{6(l-a)}}$.

Now for common Catenary we also know that

$$y = c \cosh x/c$$

For point B

$$\Rightarrow \quad y = c \cosh \frac{a}{c}$$

$$\Rightarrow \quad y = c\left[1 + \frac{a^2}{2c^2} + \frac{a^4}{4!\,c^4} + \cdots\right]$$

$$\Rightarrow \quad y = c\left[1 + \frac{a^2}{2c^2}\right]$$

$$y - c = \frac{a^2}{2c}$$

$$\text{Sag} = \frac{a^2}{2}\sqrt{\frac{6(l-a)}{a^3}}$$

$$= \frac{1}{2}\sqrt{6a(l-a)}$$

Proved.

3. Theorem of parallel axis: If the moments and products of inertia about any axis through the centre of gravity of a body are known, to find the moments and products of inertia about a parallel axis.

Let $G(\bar{x}, \bar{y}, \bar{z})$ be the centre of gravity of the body referred to ~~ray~~ rectangular axes OX, OY and OZ. Let (x, y, z) be the coordinates of a point P referred to these axes and (x', y', z') be the coordinates of the same point referred to parallel axes Gx', Gy', Gz'

so that

$$x = \bar{x} + x', \quad y = y' + \bar{y}, \quad z = z' + \bar{z}$$

If m be the mass at P then moment of inertia I of the body about Ox is given by

$$I = \Sigma m (y^2 + z^2)$$

$$= \Sigma m \left[(y' + \bar{y})^2 + (z' + \bar{z})^2\right]$$

But $\Sigma m y' = 0, \ \Sigma m z' = 0,$

Hence $I = \Sigma m (y'^2 + z'^2) + M(\bar{y}^2 + \bar{z}^2)$

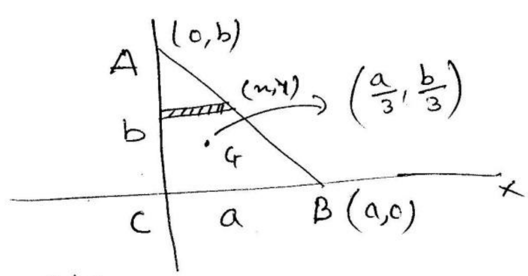
= Moment of inertia about Gx' + mass × (distance of centre of gravity from Ox)

$$\boxed{I = I_c + Md^2}$$

Similarly we can show for product of inertia.

4.

Area of $\Delta ABC = \frac{1}{2} ab$



Let m be the mass of the triangle ABC.

mass per unit area $= \dfrac{M}{\frac{1}{2}ab} = \dfrac{2M}{ab}$

Moment of inertia about $Cx = \displaystyle\int_0 (x\,dy)\left(\frac{2M}{ab}\right) y^2$

$$= \int_0^b y^2\left(1-\frac{y}{b}\right) a\left(\frac{2m}{ab}\right) dy \qquad \left| \; \frac{x}{a}+\frac{y}{b}=1 \right.$$

$$= \frac{2M}{b}\int_0^b \left(y^2-\frac{y^3}{b}\right) dy$$

$$= \frac{2M}{b}\left[\frac{y^3}{3}-\frac{y^4}{4b}\right]_0^b$$

$$= \frac{2M}{b}\left[\frac{b^3}{3}-\frac{b^3}{4}\right]$$

$$A = \frac{Mb^2}{6}$$

Similarly
Moment of inertia about y axis $B = \frac{Ma^2}{6}$

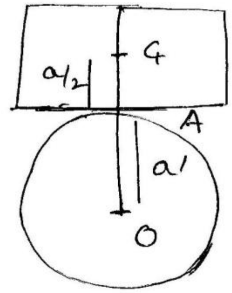
Product of inertia $F = \frac{Mab}{12}$

Therfore the angle make by the principal axis with side of the triangle $= \frac{1}{2}\tan^{-1}\frac{2F}{B-A}$

$$= \frac{1}{2}\tan^{-1}\frac{2\,\frac{Mab}{12}}{\frac{Ma^2}{6}-\frac{Mb^2}{6}}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{ab}{a^2-b^2}\right).$$

5.



Now heigh of the centre of gravity $h = \frac{a}{2}$

$r = \infty$

$R = a'$

Hence for equilibrium

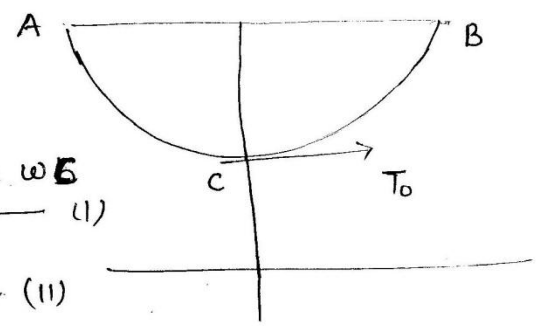
$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

$$\Rightarrow \quad \frac{2}{a} > \frac{1}{a'}$$

$$\Rightarrow \quad a' > \frac{a}{2}$$

Hence least value of $a' = \frac{a}{2}$.

6.



Length of the chain = ~~2l~~ 2S

Tension at lowest point $T_0 = wc$ ——— (I)

At B $\quad s = c\tan\psi$ ——— (II)

and tension at B $T = wy = wb$

$\Rightarrow y = b$ at B.

$\Rightarrow \quad y^2 = c^2 + s^2$

$\Rightarrow \quad b^2 = c^2 + s^2$ at B

$\Rightarrow \quad c = \sqrt{b^2 - s^2}$ ——— (III)

At B $\quad s = c \tan\psi$

$s = \sqrt{b^2 - s^2} \tan\psi$ from (III)

$\Rightarrow \quad \tan\psi = \dfrac{s}{\sqrt{b^2 - s^2}}$ ——— (IV)

$\Rightarrow \quad \sec\psi = \dfrac{b}{\sqrt{b^2 - s^2}}$ ——— (V)

Now $\quad$ span $= AB$

$= 2x$

$= 2c \log(\sec\psi + \tan\psi)$

Putting values from (III), (IV) and (V), we get

$$\text{span} = \sqrt{b^2 - s^2} \log\left(\frac{b+s}{b-s}\right).$$

Ans

7- Motion of a compound pendulum: A compound pendulum is one where the rod is not massless, and may have extended size, that is, an arbitrarily shaped rigid body swinging by a pivot. In case pendulum periods depends on its moment of inertia I around the pivot point.

The equation of torque is given by $\tau = I\alpha$, —(I)

where α is the angular acceleration, τ is the torque.

Since torque is generated by gravity so

$$\tau = -mgl \sin\theta \quad \text{———} \quad (II)$$

For small angle approx. $\sin\theta \cong \theta$

From (II) & (I), we have

$$\alpha \simeq \frac{mgl\,\theta}{I}$$

This gives a period of

$$\boxed{\tau = 2\pi\sqrt{\frac{I}{mgl}}}$$

Ans

8 Stability of a body with one degree of freedom:

Let θ be the geometrical quantity such that

$$W = f(\theta) \quad \text{———} \quad (i)$$

Let F be the resultant force acting on the body and δs be the small displacement in the direction of F. then

$$\delta W = F \delta s \quad \text{———} \quad (ii)$$

From (i), we can write

$$f'(\theta)\, \delta\theta = F \delta s$$

$$\Rightarrow \quad f'(\theta) \frac{d\theta}{ds} = F \quad \text{———} \quad (iii)$$

For equilibrium $\quad F 0$

$$\Rightarrow \quad f'(\theta) \frac{d\theta}{ds} = 0$$

$$\Rightarrow \quad f'(\theta) = 0 \quad \text{———} \quad (iv)$$

In the case of stable equilibrium, $\frac{dF}{ds} < 0$

Now (iii) implies

$$f''(\theta_1) \left(\frac{d\theta}{ds}\right)^2 + f'(\theta_1) \frac{d^2\theta}{ds^2} < 0,$$

since $f'(\theta_1) = 0$

$$\therefore \quad f''(\theta_1) < 0$$

Thus for stable at $\theta = \theta_1$

(i) $f'(\theta_1) = 0$

(ii) $f''(\theta_1) < 0$

Similarly it may be shown that if W is a minimum i.e $f''(\theta_1) > 0$, the equilibrium is stable.

(Dr. B.B. Chaturvedi)